Chapter 3: Counting Reference Sheet

Bit Strings Definitions

A **bit string** is a string of binary digits.

An n -**bit string** is a bit string of length n. The **weight** of a bit string is the number of 1's it contains. B_k^n is the set of all n -bit strings of weight k.

Binomial Theorem

The n th row of Pascal's triangle gives the coefficients of the expansion of $(x+y)^n$. That is, for any positive integer n,

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n,$$

so the coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$.

Combining Outcomes

We speak of counting the number of **outcomes** that can result from a given **event**. In terms of sets, an outcome is an element from the set that satisfies the requirements defined in the event. So counting outcomes is finding the size of the subset of elements meeting the event requirements.

Two Ways to Think of Combining Outcomes

There are two ways we can think of combining outcomes:

1. We can combine the *sets* of outcomes.
2. We can combine the *outcomes* in the sets.

Sum Principle

If an event A results in m outcomes, and event B results in n *disjoint* outcomes, then the event $A \text{ or } B$ results in $m+n$ outcomes.

Product Principle

If event A can occur in m ways, and each possibility for A allows for exactly n ways for event B, then the event $A \text{ and } B$ can occur in $m \cdot n$ ways.

Additive Principle (with sets)

If event A can occur in m ways, and event B can occur in n *disjoint* ways, then the event A or B can occur in $m + n$ ways.

Stated using set notation, we have:

Given two sets A and B, if $A \cap B = \emptyset$, then

$$|A \cup B| = |A| + |B|$$

Note that this is a restatement of the previous Sum Principle using set notation.

Multiplicative Principle

If event A can occur in m ways, and each possibility for A allows exactly n ways for event B, then the event A and B can occur in $m \cdot n$ ways.

Stated using set notation, given two sets A and B we have

$$|A \times B| = |A| \cdot |B|$$

Again, this is a restatement of the Product Principle using sets.

Cartesian Product

Given sets A and B, we can form the set

$$A \times B = \{(x, y) : x \in A \land y \in B\}$$

to be the set of all ordered pairs (x, y) where x is an element of A and y is an element of B. We call $A \times B$ the **Cartesian product** of A and B.

Cardinality of a Union of Two Sets

For any finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Cardinality of a Union of Three Sets

For any finite sets A, B, and C,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Principle of Inclusion/Exclusion (PIE)

This principle can be applied to any number of sets, but it becomes ever more complicated as the number of sets increases. For four sets, we have:

$$\begin{gathered}
|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| \\
-|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
+|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\
-|A \cap B \cap C \cap D|.
\end{gathered}$$

Permutation

A **permutation** is a (possible) rearrangement of objects. For example, the six permutations of the letters a, b, and c are:

> abc, acb, bac, bca, cab, cba

Permutations of n elements

There are

$$n! = n \cdot (n-1) \cdot (n-2) \cdot ... \cdot 2 \cdot 1$$

permutations of n (distinct) elements.

k-permutations **of n elements**

$P(n,k)$ is the number of k-permutations **of n elements**, the number of ways to arrange k objects chosen from n distinct objects.

$$P(n,k) = \frac{n!}{(n-k)!} = n(n-1)(n-2)...(n-(k-1))$$

> Note that when $n = k$, we have $P(n,n) = \frac{n!}{(n-n)!} = n!$, since we defined $0! = 1$.

Closed formula for $\binom{n}{k}$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)...(n-(k-1))}{k(k-1)(k-2)...1}$$

Multisets

A **multiset** is an unordered collection of elements, each of which can appear any number of times. The number of times an element appears is called its **multiplicity**.

Multisets are written using the same noation as sets: a comma-separated list in braces, such as $\{1, 2, 2, 5\}$.

"Sticks and Stones" Formulas

Putting n objects into k bins can be done in this many ways:

$$\binom{n+k-1}{k-1} \text{ or } \binom{n+k-1}{n}.$$

If each bin must contain *at least one* item, then we have this many ways:

$$\binom{n-1}{k-1}.$$

Counting Outcomes Table

	Order matters	Order doesn't matter
Repeats	n^k	$\binom{n+k-1}{k-1}$
No Repeats	$P(n,k)$	$\binom{n}{k}$

Probability Definition

Suppose a random experiment has sample space S. The **probability** of an event E is the number of outcomes in E divided by the number of outcomes in S. We write this as $P(E) = \frac{|E|}{|S|}$.

Probability of the Complement Theorem

The probability of the complement of an event E is

$$P(\bar{E}) = 1 - P(E).$$

Probability of Disjoint Events Theorem

Suppose A and B are two **disjoint** events. Then the probability of either A or B happening is,

$$P(A \cup B) = P(A) + P(B).$$

If A and B are not disjoint, then the probability of A or B occuring is,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Independent Events Definition

Given two events A and B, we say they are **independent** provided the probability of both events happening is the product of the probabilities of each event happening:

$$P(A \cap B) = P(A)P(B).$$

Conditional Probability Definition

Given two events A and B, the **conditional probability** of A given B is,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$